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Some invariants for equivalent G-algebras

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Abstract

In an earlier paper (Clifford theory with Schur indices, *J. Algebra* 170 (1994) 661–677), the author introduced a generalization of the Brauer–Wall group. It is defined for any given finite group G and any field F of characteristic 0. Each element of this generalized Brauer–Wall group is an equivalence class of *central simple* G-algebras. He showed that given a finite group H with a normal subgroup N such that $H/N \simeq G$, and an irreducible character χ of H, there corresponds naturally an element of this generalized Brauer–Wall group, and that this element alone controls the Clifford theory (including Schur indices) of χ with respect to N. The present paper studies some invariants for G-algebras which are preserved under equivalence of G-algebras in the above sense. These invariants are the basis of a characterization of each equivalence class of central simple G-algebras in some cases, as is described in a forthcoming paper of the author. The present paper also includes a brief comparison of this generalization of the Brauer–Wall group with others that have appeared in the literature.

0. Introduction

In [10], the set $S(G,F)$ is introduced for each finite group G and field F, see also below. $S(G,F)$ is a generalization of the Brauer group of a field, except that no natural group structure is present in $S(G,F)$. Its elements are equivalence classes of central simple G-algebras. The concepts of central simple G-algebra and of equivalence generalize their homonyms for algebras. The elements of $S(G,F)$ control the Clifford theory of every group H_1 containing a normal subgroup H_2 such that $H_1/H_2 \simeq G$. Namely, to every irreducible character χ of H_1, it is shown in [10] how to associate an element $[[\chi]]$ of $S(G, \mathbb{Q}(\chi|_{H_2}))$, which contains the information about how characters extend or are induced among subgroups of H_1 which contain H_2, their relative degrees, their fields of values and their Schur indices. If M is a χ-quasihomogeneous module for H_1 over $\mathbb{Q}$, a condition that one only needs to verify in terms of the character afforded by M, then $\mathrm{End}_{\mathbb{Q}H_2}(M)$ has a natural structure as a G-algebra over

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$\mathbb{Q}(\chi|_{H_2})$, and is a representative of $[[\chi]] \in S(G, \mathbb{Q}(\chi_{|H_2}))$, see [10] for details. What aspects of the structure of $\mathrm{End}_{\mathbb{Q}H_2}(M)$ are the relevant ones that describe its class in $S(G, F)$? In the present paper we begin to answer this question by defining some structures for every central simple G-algebra which are invariants (up to isomorphism) of their class.

If $G = 1$, then $S(G, F)$ is just the set of elements of the Brauer group of F. If $|G| = 2$ and the characteristic of F is not 2, then $S(G, F)$ is easily seen to be equivalent to the set of elements of the Brauer–Wall group of F, see [7] for the definition and properties of the Brauer–Wall group. The elements of the Brauer–Wall group are really $\mathbb{Z}_2$-graded algebras over F, but, in this case, it is easy to see that a $\mathbb{Z}_2$-grading structure on an algebra over F is equivalent to a G-action on the same algebra. A full set of invariants is available in this case. Each element of the Brauer–Wall group of F is characterized by three invariants: a parity, an element of $F^\times/(F^\times)^2$, and an element of the Brauer group $\mathrm{Br}(F)$ of F. The situation for general G is much more complicated.

In the present paper, we begin to work toward a characterization of the classes of central simple G-algebras A, for an arbitrary finite group G. We see that to every class is associated a subgroup of G, the inertia group I, that is readily obtained from any one representative of the class. More precisely, the inertia subgroup is uniquely determined up to conjugacy in G by the equivalence class of A. In the case $|G| = 2$ and $\mathrm{Char}(F) \neq 2$, this inertia group determines exactly the parity of A as an element of the Brauer–Wall group.

Furthermore, for each inertia subgroup I, we construct a structure, the I-centroid of A, which also depends only on the equivalence class of A. This structure, which we denote by $\Delta(A, I)$, or $\Delta(A)$ if I is a normal subgroup of G, is an algebra of dimension $|N_G(I)|$ over F, and has both an I-grading and an action by $N_G(I)$, with certain compatibility conditions (see Definition 3.1). In the case $|G| = 2$ and $\mathrm{Char}(F) \neq 2$, the structure of $\Delta(A)$ is determined, up to isomorphism, exactly by I and an element of $F^\times/(F^\times)^2$, see Remark 3.2. In this particular case the invariant $\Delta(A)$ plays the role of the first two invariants (parity and an element of $F^\times/(F^\times)^2$) mentioned for the Brauer–Wall group.

Let A be a central simple G-algebra over F, and K be an extension field of F. Then, $A \otimes_F K$ is a central simple G-algebra over K in a natural way, and this process respects equivalence, see Proposition 1.9. The invariants defined in this paper behave well with respect to this process. Namely, if I is an inertia group of A, then I is also an inertia group of $A \otimes_F K$, see Proposition 2.7. Furthermore,

$$\Delta(A \otimes_F K, I) \simeq \Delta(A, I) \otimes_F K,$$

see Proposition 3.7.

The results of the present paper will be used to characterize all the equivalence classes of central simple G-algebras in certain cases. In a forthcoming article, we will introduce the concept of a 'complement' R for a given centroid Δ. Then, we will see that the isomorphism types of centroids with complement R are classified, up to isomorphism, by two invariants: Δ_1 and an element $\eta(\Delta, R) \in H^2(I, F^\times)$. All the possibilities for these invariants will be characterized. Furthermore, for each such Δ, a subgroup of $\mathrm{Br}(F)$, depending only on R and Δ_1, will be defined, and a bijection between the equivalence classes of central simple G-algebras with centroid isomorphic to Δ and the cosets of $\mathrm{Br}(F)$ under this subgroup will be established. These considerations will be enough to classify, in particular, all central simple G-algebras, up to equivalence, for many abelian G. This will be the case, for example, if F is a real field and G has either cyclic or elementary abelian Sylow 2-subgroup. The classification will also be complete if F is any number field and G is cyclic.

Since Wall's paper [11] of 1964, various generalizations of the Brauer–Wall group involving grading or acting groups larger than $\mathbb{Z}_2$ have been proposed. In fact a substantial literature has emerged on the subject, of which we mention only [1–6, 8] and invite the interested reader to check also the references in these papers. The results concern primarily the case where G is abelian. The most general group proposed so far is one introduced by Long [8] that contains all the others as specific subgroups. In the Brauer–Long group we consider G-graded G-algebras (or more generally dimodule algebras for some Hopf algebra) with the condition that they be G-Azumaya. The structure of the Brauer–Long group in the case of G-graded G-algebras is described in [2]. However, in the case that we need to consider for our applications, namely trivially graded simple G-algebras over a field F, the condition that A is G-Azumaya boils down to $Z(A) = F$. Although the classes of algebras with this condition do form a group whose structure is easily described, the condition is far too restrictive for the applications we have in mind.

If F has enough roots of unity and still for abelian G a different point of view is suggested by the work of Knus [6]. Let $\phi: G \times G \to F^\times$ be a fixed bimultiplicative map. Then if A is a G-algebra, one can use ϕ to give A also a G-grading in a non-trivial way. Viewing each G-algebra as a G-graded G-algebra in this way we obtain a subgroup of the Brauer–Long group which depends on ϕ. This subgroup has been studied in its own right [3, 4, 6]. Even though using this approach we may study certain G-algebras A with $Z(A) \neq F$, many G-algebras of interest have to be excluded, see Remark 1.8, for more details. Hence this approach is not suitable for the applications we have in mind.

In the point of view adopted in the present paper, we agree to examine equivalence among all central simple G-algebras over a field F. The price we pay is that the resulting set of classes is no longer a group.

1. Basic definitions and results

Throughout, F will denote a field and G a finite group. We begin by recalling some of the basic definitions and results from [10].

Definitions 1.1. A *G-algebra* is a finite-dimensional associative algebra A over F with $1 \neq 0$ together with a homomorphism

$$\psi : G \longrightarrow \mathrm{Aut}(A),$$

from G into the group of F-algebra automorphisms of A. We say that G acts on A and we use the reverse exponential notation for the action, so we write

$$\psi(g)(a) = {}^g a$$

for $g \in G$, $a \in A$. Similarly, we call centralizer the set of elements of A fixed by some given elements of G, and we write, for example,

$$C_A(g) = \{a \in A \mid {}^g a = a\}$$

for $g \in G$.

We say that a G-algebra A is *simple* if it does not have any G-invariant (two sided) ideals other than A and 0. We say that a G-algebra A is *central* if $C_{Z(A)}(G) = F$, where $Z(A)$ is the center of A.

Two G-algebras A and B are *isomorphic* if there exists a map

$$\varphi : A \longrightarrow B,$$

which is an algebra isomorphism and such that for every $a \in A$, $g \in G$,

$$\varphi({}^g a) = {}^g \varphi(a).$$

Note however that if A is a G-algebra and we say that A is a simple algebra or that A is a central algebra we will mean in each case as an algebra (ignoring the action of G). This is in contrast with saying A is a simple G-algebra or central G-algebra, which means as just defined.

Lemma 1.2. *Let A be a simple G-algebra. Then, A is a semisimple algebra. In particular, as an algebra*

$$A = A_1 \oplus \cdots \oplus A_n,$$

where $A_1, \ldots, A_n$ are simple ideals of the algebra A, $A_i A_j = 0$ for $i \neq j$. Furthermore, G permutes the $\{A_1, \ldots, A_n\}$ transitively.

Proof. This is Lemma 1.3 in [10]. □

If A and B are G-algebras, then $A \otimes B$ (the tensor product always over F) is a G-algebra in a natural way.

Lemma 1.3. *Let A and B be simple G-algebras and assume that B is central simple as an algebra. Then, $A \otimes B$ is a simple G-algebra. If A is a central G-algebra, then $A \otimes B$ is a central simple G-algebra.*

Proof. This is Lemma 1.4 in [10]. □

We are now ready to define our equivalence relation.

Definition 1.4. We will say that a G-algebra is *trivial* if it is G-isomorphic to a G-algebra E obtained in the following way. Let $N \neq 0$ be some FG-module and set $E = \mathrm{End}_F(N)$. Then, E is a central simple algebra. The module structure of N provides a homomorphism

$$\varphi : G \longrightarrow E^{\times}$$

and, for $g \in G$, $a \in E$, we set ${}^g a = \varphi(g)\, a\, \varphi(g)^{-1}$. This provides E with the structure of a central simple G-algebra.

Definition 1.5. If A and B are central simple G-algebras, we say that A and B are *equivalent* and we write $A \sim B$ if and only if there exist trivial G-algebras E and E' such that

$$A \otimes E \simeq B \otimes E'$$

as G-algebras.

Proposition 1.6. *The relation $\sim$ is an equivalence relation on the class of central simple G-algebras.*

Proof. This is Proposition 1.7 in [10]. □

Definition 1.7. Let G be a finite group and F a field. We denote by $S(G,F)$ the collection of equivalence classes of central simple G-algebras over F. If A is a central simple G-algebra over F we denote by $[A]$ the element of $S(G,F)$ that A represents.

Remark 1.8. If G is abelian and F has a primitive $\exp(G)$th root of 1 (where $\exp(G)$ is the exponent of the finite group G), then every G-algebra A becomes a $\hat{G}$-graded algebra in a natural way and vice versa. Here $\hat{G}$ is the set of group homomorphisms from G to $F^{\times}$. If $\lambda \in \hat{G}$, then

$$A_\lambda = \{a \in A \mid {}^g a = \lambda(g)a \text{ for all } g \in G\},$$

and it is a straight forward exercise to show that this provides A with the structure of a $\hat{G}$-graded algebra. The reverse correspondence from $\hat{G}$-graded algebras to G-algebras is likewise straightforward. So one might ask what, in this case, is the relationship between our definition of A being a central simple G-algebra and A being a central simple $\hat{G}$-graded algebra, for example in the sense of [3,4]. The answer is that every central simple $\hat{G}$-graded algebra in the sense of [3,4] is a central simple G-algebra in our sense but not vice versa. The key is in the definition of the *graded-center* of A in [3,4]. For it, we need to pick some bilinear form

$$f: \hat{G} \times \hat{G} \to F^{\times}$$

and, the graded center of A is defined in [3,4] as the graded subalgebra $\hat{Z}(A)$ of A spanned by all the homogeneous elements $a \in A_{\varphi}$ say, such that

$$ax = f(\varphi, \mu)xa$$

for every $\mu \in \hat{G}$ and every $x \in A_{\mu}$. Then A is said to be *graded-central* if $\hat{Z}(A) = F$. A graded F-simple $\hat{G}$-graded algebra A is then a graded simple central algebra if A is also graded-central. It is easy to see that if A is a $\hat{G}$-graded graded central simple algebra, then its corresponding G-algebra A is central simple in our sense as a G-algebra. However, the converse is definitely not the case. For an easy example, let $G = \langle g \rangle$ be a cyclic group of order 4 and let $F = \mathbb{Q}(\mathrm{i})$, where $\mathrm{i}^2 = -1$. Let $A = F(\sqrt{3})$. Then A is a Galois field extension of F of degree 2. Let G act on A via the non-trivial homomorphism $G \to \mathrm{Gal}(A/F)$ (with kernel of order 2). Clearly A is a central simple G-algebra in our sense. However $\hat{G}$ is a cyclic group of order 4 and $\sqrt{3}$ is graded by the element φ of order 2 in $\hat{G}$. Let ϕ be a generator for $\hat{G}$. Then, no matter which f we choose we will have

$$f(\varphi, \varphi) = f(\phi^2, \phi^2) = f(\phi, \phi^4) = 1.$$

From this it follows that $\sqrt{3} \in \hat{Z}(A) = A$. So A is not graded central simple in the sense of [3,4].

If K is an extension field of F, then there is a natural map $\mathrm{S}(G, F) \to \mathrm{S}(G, K)$, which we now describe. We will call this map extension of scalars.

Proposition 1.9. *Let G be a finite group, F a field and K an extension field of F. If A is a central simple G-algebra over F, then $A \otimes_F F$ is a central simple G-algebra over K. Furthermore, this map from central simple G-algebras over F to central simple G-algebras over K induces a map $S(G, F) \to S(G, K)$.*

Proof. Set $\bar{A} = A \otimes_F K$. By Lemma 1.2, A is a semisimple algebra over F. It follows from Lemma 2.1 and Theorem 2.2, that $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a finite-dimensional simple algebra over F and $\mathrm{Z}(A_i)/F$ is a separable field extension.

Hence, A is a separable algebra over F. Therefore $\bar{A}$ is a semisimple algebra over K. Since A is a central G-algebra $\dim_F(C_{Z(A)}(G)) = 1$. Hence $\dim_K(C_{Z(\bar{A})}(G)) = 1$. This means that $C_{Z(\bar{A})}(G) = K$. Let $e_1, \dots, e_n$ be the primitive central idempotents of $\bar{A}$. These are permuted by the action of G. Since the dimension over K of the centralizer of G in $Z(\bar{A})$ is one, it follows that G can only have one orbit in its action on $\{e_1, \dots, e_n\}$. Now the minimal ideals of $\bar{A}$ are exactly $e_1\bar{A}, \dots, e_n\bar{A}$. Therefore, G acts transitively on the minimal ideals of $\bar{A}$ and $\bar{A}$ is the sum of its minimal ideals. Hence, any non-zero G-invariant ideal I of $\bar{A}$ must contain one (hence all) minimal ideals of $\bar{A}$, and must then be $\bar{A}$. This shows that $\bar{A}$ is a central simple G-algebra over K.

Suppose B is a central simple G-algebra over F and B is equivalent to A. Set $\bar{B} = B \otimes_F K$. There exists trivial G-algebras T, T' such that $A \otimes_F T \simeq B \otimes_F T'$ as G-algebras. It follows immediately from Definition 1.4 that $T \otimes_F K$ and $T' \otimes_F K$ are trivial G-algebras over K. Furthermore,

$$\bar{A} \otimes_K (T \otimes_F K) \simeq A \otimes_F T \otimes_F K \simeq B \otimes_F T' \otimes_F K \simeq \bar{B} \otimes_K (T' \otimes_F K).$$

This means that $\bar{A} \sim \bar{B}$ as central simple G-algebras over K. Hence, $A \to A \otimes_F K$ does induce a map $S(G,F) \to S(G,K)$. This concludes the proof of the proposition. □

2. The center and the inertia group

The first invariant that we discuss is the center, which turns out to be a central simple G-algebra in its own right.

Lemma 2.1. *Let A be a commutative central simple G-algebra. Let $e_1, \dots, e_\alpha$ be the primitive idempotents of A, and set $K_i = e_iA$ for $i = 1, \dots, \alpha$. Set $I = C_G(K_1)$. Then*

(a) $\dim_F(A) = [G:I]$.

(b) *For $i = 1, \dots, \alpha$, K_i is a field and K_i/e_iF is a Galois extension with Galois group $C_G(e_i)/C_G(K_i)$.*

(c) *G acts transitively on $\{e_1, \dots, e_\alpha\}$.*

Proof. By Lemma 1.2, A is a semisimple algebra and $K_1, \dots, K_\alpha$ are its minimal ideals, on which G acts transitively. Hence (c) holds. $C_G(e_i)/C_G(K_i)$ acts as automorphisms of K_i fixing e_iF. Now suppose $x \in K_i$ is fixed by every element of $C_G(e_i)$. Let $h_1, \dots, h_\alpha \in G$ be representatives for the left cosets of $C_G(e_i)$ in G (by (c) $[G:C_G(e_i)] = \alpha$).

Set

$$y = \sum_{i=1}^{\alpha} {}^{h_i}x.$$

It is easy to see that $y \in C_A(G)$. Since A is a central G-algebra, it follows that $y \in F$. Since the summands of y belong to different K_i's and $A = K_1 \oplus \cdots \oplus K_\alpha$, we have

$x = e_i y$ and $x \in e_i F$. Hence, b holds. In particular,

$$\dim_F(K_i) = [C_G(e_i) : C_G(K_i)].$$

Since $\alpha = [G : C_G(e_i)]$ and $A = K_1 \oplus \cdots \oplus K_\alpha$, it follows that

$$\dim_F(A) = [G : I],$$

i.e. that a holds. This completes the proof of the lemma. □

Theorem 2.2. *Let A be a central simple G-algebra. Then, $Z(A)$, the center of A, is a commutative central simple G-algebra. Furthermore, if B is a central simple G-algebra and A and B are equivalent, then $Z(A)$ and $Z(B)$ are isomorphic as G-algebras.*

Proof. Since A is a central simple G-algebra, by Lemma 1.2,

$$A = A_1 \oplus \cdots \oplus A_n,$$

where $A_1, \ldots, A_n$ are simple ideals of the algebra A and G acts transitively on $\{A_1, \ldots, A_n\}$. Since $A_1, \ldots, A_n$ are simple algebras, $Z(A_1), \ldots, Z(A_n)$ are fields, and G also acts transitively on $\{Z(A_1), \ldots, Z(A_n)\}$. But now

$$Z(A) = Z(A_1) \oplus \cdots \oplus Z(A_n)$$

and any non-zero ideal of $Z(A)$ must contain one of $Z(A_1), \ldots, Z(A_n)$. Hence, if this ideal is G-invariant, it must be $Z(A)$. It follows that $Z(A)$ is a commutative central simple G-algebra.

Since A and B are equivalent, there exist trivial G-algebras T and T' such that

$$A \otimes T \simeq B \otimes T'$$

as G-algebras. But the center of $A \otimes T$ is G-isomorphic to $Z(A)$ as $Z(T) = F$ and similarly for B. Hence, the restriction of the isomorphism $A \otimes T \simeq B \otimes T'$ to the centers yields the desired G-isomorphism between $Z(A)$ and $Z(B)$. This concludes the proof of the theorem. □

Remarks 2.3. Let K be an extension field of F. Let $[A] \in S(G,F)$ and let $[\bar{A}] \in S(G,K)$ be its corresponding extension of scalars, see Proposition 1.9. Let Z be the center of $[A]$. By Theorem 2.2, Z is a well-defined (up to isomorphism) commutative central simple G-algebra over F. The center of $[\bar{A}]$ is just $Z \otimes_F K$.

Definition 2.4. Let A be a central simple G-algebra. Let e be a primitive central idempotent of the algebra A and let $K = eZ(A)$. Set $I = C_G(K)$. Then, we say that I is an *inertia* group of the G-algebra A.

Proposition 2.5. *Let A be a central simple G-algebra. Then, the set of inertia groups of A is a conjugacy class of subgroups of G, and depends only on the equivalence class of A.*

Proof. By Theorem 2.2, $Z(A)$ is a central simple G-algebra and depends only on the equivalence class of A. By Lemma 2.1, all the primitive central idempotents of $Z(A)$ are conjugate under the action of G. Hence, the possible inertia groups are also conjugate under the action of G. □

Lemma 2.6. *Let K/F be a finite Galois extension of fields with Galois group G. Let L be an extension field of F. Set $\bar{K} = K \otimes_F L$. Let e be a primitive idempotent of $\bar{K}$. Then, $C_G(e)$ acts faithfully on $e\bar{K}$.*

Proof. K is a central simple G-algebra over F. It follows, by Proposition 1.9, that $\bar{K}$ is a central simple G-algebra over L. Since $\dim_L(\bar{K}) = \dim_F(K) = |G|$, it follows from Lemma 2.1.a that $C_G(e\bar{K}) = 1$. Hence, the lemma holds. □

Proposition 2.7. *Let K be an extension field of F. Let $[A] \in S(G,F)$, and let $[\bar{A}] \in S(G,K)$ be its corresponding extension of scalars, see Proposition 1.9. The inertia groups of $[A]$ are the inertia groups of $[\bar{A}]$.*

Proof. Let Z be the center of $[A]$. Then, Z is a central simple G-algebra. The inertia groups of $[A]$ are obtained as follows. Let e be any primitive idempotent of Z. Then, $I = C_G(eZ)$ is an inertia group of $[A]$, and all the inertia groups are obtained in this way. By Lemma 2.1, eZ/eF is a finite Galois extension of fields with Galois group $C_G(e)/I$. By Remark 2.3, the center of $[\bar{A}]$ is just $\bar{Z} = Z \otimes_F K$. Let $\bar{e}$ be any primitive idempotent of $Z \otimes_F K$ contained in e. By Lemma 2.6, $C_G(\bar{e})/I$ acts faithfully on $\bar{e}\bar{Z}$. Hence, $I = C_G(\bar{e}\bar{Z})$ and I is also an inertia group of $[\bar{A}]$. As the inertia groups form a conjugacy class of subgroups in G, the proposition follows. □

3. The I-centroid

In this section we study an object which we call the I-centroid of A. It turns out to be a *central fully I-graded H-algebra*, where $H = N_G(I)$ is the normalizer of an inertia group of A. We begin by defining this resulting structure.

Definition 3.1. Let F be a field, H a finite group and I a normal subgroup of H. A structure Δ is said to be a *central fully I-graded H-algebra* over F if the following hold.

(a) Δ is a fully I-graded algebra over F, that is we are given

$$\Delta = \bigoplus_{g \in I} \Delta_g,$$

where this is a direct sum decomposition of vector spaces over F, and the algebra structure of Δ is such that $\Delta_g \Delta_h = \Delta_{gh}$ as sets for all $g, h \in I$.

(b) We are also given an action of H on Δ in such a way that, the action preserves the algebra structure of Δ, for every $h \in I$ and $a \in \Delta_h$, $b \in \Delta$, we have $ab = {}^h ba$, and for all $h \in H$, $g \in I$, $a \in \Delta_g$, we have ${}^h a \in \Delta_{hgh^{-1}}$.

(c) Finally, Δ_1 is a finite dimensional semisimple commutative algebra, the centralizer of the action of H on Δ_1 is I and $C_{\Delta_1}(H) = F.1 = F$.

Remark 3.2. In the case where $|G| = 2$ and the characteristic of F is not two, the structure of Δ is determined, up to isomorphism, exactly by I and an element of $F^\times/(F^\times)^2$, as follows. Let $G = \{1, g\}$.

Suppose, first, that $I = G$. Then, by c, G acts trivially on Δ_1 and $\Delta_1 = F.1$. By a, there is some invertible element $\lambda \in \Delta_g$. Left multiplication by λ provides an F-vector space isomorphism from Δ_1 to Δ_g, so that Δ is two dimensional and $\Delta_g = \lambda\Delta_1$. Now $\lambda^2 \in F^\times$. Furthermore, if $\mu \in \Delta_g$ and $\mu \neq 0$, then $\mu = a\lambda$ for some $a \in F^\times$, so that $\mu^2 = a^2\lambda^2$. Hence, λ^2, up to squares in $F^\times$, depends only on Δ. Now the G-graded algebra structure of Δ is determined, up to isomorphism, by λ^2, up to squares. In every case, Δ is commutative, so, by b, for every $b \in \Delta$, $b = \lambda b \lambda^{-1} = {}^g b$, and, G acts trivially on Δ. Hence, the isomorphism type of Δ is exactly determined by λ^2 up to squares in $F^\times$, in this case.

Now suppose that $I = 1$. By (c) $\Delta = \Delta_1$ is a finite-dimensional semisimple commutative algebra on which G acts faithfully, and with $C_\Delta(G) = F$. Let $\Delta^- = \{a \in \Delta: {}^g a = -a\}$. Since the characteristic of F is not two, $\Delta = F \oplus \Delta^-$, as vector spaces. Because G acts faithfully on Δ, $\Delta^- \neq 0$. Let $\lambda \in \Delta^-$ be non-zero. Since Δ is commutative and semisimple, $\lambda^2 \neq 0$. Clearly, $\lambda^2 \in C_\Delta(G) = F$. It follows that λ is invertible and that $\Delta^- = \lambda F$. In particular, Δ is two dimensional. As in the previous case, $\lambda^2 \in F^\times$ is determined, up to squares in $F^\times$, by Δ. The G-algebra structure of Δ is determined, up to isomorphisms, by λ^2, up to squares. Since the grading on Δ is trivial, the isomorphism type of Δ is exactly determined by λ^2 up to squares in $F^\times$, also in this case.

That the dimension of Δ over F is $|H|$ is not a special property of the case $|H| = 2$. It holds always, as our next lemma shows.

Lemma 3.3. *Let F be a field, H a finite group and I a normal subgroup of H. Let Δ be a central fully I-graded H-algebra over F. Then,* $\dim_F(\Delta) = |H|$.

Proof. Since Δ is fully I-graded, for each $g \in I$, there is some invertible element of Δ in Δ_g. Hence, left multiplication by such an element proves that $\Delta_1 \simeq \Delta_g$ as vector spaces over F. Hence, $\dim_F(\Delta) = |I|\dim_F(\Delta_1)$. Now Δ_1 is a semisimple commutative H-algebra and $C_{\Delta_1}(H) = F$. Now it follows, as in the proof of Proposition 1.9, that H must act transitively on the primitive idempotents of Δ_1. Hence, Δ_1 is a commutative central simple H-algebra. By Lemma 2.1.a, it follows that $\dim_F(\Delta_1) = [H:I]$. Hence, $\dim_F(\Delta) = |H|$, as required. $\square$

Definition 3.4. Let F be a field, G be a finite group and A a central simple G-algebra over F. Let I be an inertial group for A, and set $H = N_G(I)$. The *I-centroid* of A, or simply the *centroid* of A if I is normal in G, is the following structure $\Delta = \Delta(A, I)$, also denoted $\Delta(A)$ if I is a normal subgroup of G. Let $\mathscr{I}$ be the set of all primitive central idempotents f of A such that $C_G(fZ(A)) = I$. By Definition 2.4, $\mathscr{I} \neq \emptyset$. Set $e = \sum_{f \in \mathscr{I}} f$. Then, e is a central H-invariant idempotent of A. (In particular, if I is a normal subgroup of G, then, $e = 1$.) For each $g \in I$, we let

$$\Delta_g = \{a \in Ae\colon \text{for all } b \in Ae,\ ab = {}^g ba\}.$$

We define Δ to be the abstract direct sum $\Delta = \bigoplus_{g \in I} \Delta_g$. Each Δ_g is a vector space over F, and we give Δ its natural vector space structure as a direct sum. A direct computation shows that, if $h, g \in I$ and $a \in \Delta_h$, $b \in \Delta_g$, then the product ab in A is an element of Δ_{hg}. We define the product in Δ by extending this product bilinearly. Finally, another computation shows that, if $h \in H$, $g \in I$, $a \in \Delta_g$, then the result of the action of h on a in A, namely ${}^h a$, is an element of $\Delta_{hgh^{-1}}$. We extend this action linearly to an action on Δ.

Proposition 3.5. *Let F be a field, G be a finite group and A a central simple G-algebra over F. Let I be an inertia group for A, and set $H = N_G(I)$. The I-centroid $\Delta = \Delta(A, I)$ of A is a well-defined central fully I-graded H-algebra over F. Furthermore,* $\dim_F(\Delta) = |H|$.

Proof. It follows immediately from the definition that $\Delta_1 = Z(Ae) = eZ(A)$. By Lemma 1.2, A is a semisimple algebra, so Δ_1 is a finite dimensional semisimple commutative algebra. As $\mathscr{I} \neq \emptyset$, and $\Delta_1 = \bigoplus_{f \in \mathscr{I}} fZ(A)$, the definition of $\mathscr{I}$ implies that the centralizer of the action of H on Δ_1 is I. Suppose $z \in C_{\Delta_1}(H)$. Let $g_1, \dots, g_n$ be representatives for the left cosets of H in G. Set $z' = \sum_{i=1}^{n} {}^{g_i} z$. Then, the element $z' \in Z(A)$ does not depend on the representatives $g_1, \dots, g_n$ chosen, and, $z' \in C_{Z(A)}(G) = F.1$. If $f, f' \in \mathscr{I}$ and $g \in G$ but $g \notin H$, then ${}^g f f' = 0$. Hence, $z'e = z$ and it follows that $z \in Fe$. Hence, c of Definition 3.1 holds. The definition of Δ_h implies that for every $h \in I$ and $a \in \Delta_h$, $b \in \Delta$, we have $ab = {}^h ba$ whenever b is homogeneous, and, therefore, we have $ab = {}^h ba$ for every $b \in \Delta$. Hence, b of Definition 3.1 also holds.

We now show that Δ is fully graded. Let $h \in I$. For each $f \in \mathscr{I}$, Af is a simple algebra, and h acts on it by algebra automorphisms. Since h acts trivially on $fZ(A) = Z(Af)$, by the Skolem–Noether Theorem, there exists an element $\alpha_f \in Af$ which is invertible in Af and such that, for all $b \in Af$, ${}^h b = \alpha_f b (\alpha_f)^{-1}$, where the inverse here is taken in Af. We may set $a_h = \sum_{f \in \mathscr{I}} \alpha_f$. Then, a_h is invertible in Ae and, for every $b \in Ae$, ${}^h b = a_h b a_h^{-1}$, where this time the inverse is taken in Ae. Now $a_h \in \Delta_h$ and $a_h^{-1} \in \Delta_{h^{-1}}$. Left multiplication by a_h provides an F-linear bijection from Δ_1 to Δ_h. In particular, $a_h \Delta_1 = \Delta_h$. Since Δ_1 has an identity, $\Delta_1^2 = \Delta_1$. Hence, if h, $g \in I$, then $\Delta_h \Delta_g = a_h \Delta_1 a_g \Delta_1 = a_h a_g \Delta_1$, because Δ_1 is in the center of Δ. But $a_h a_g$ is an invertible element of Δ_{hg}, and it follows that $\Delta_h \Delta_g = \Delta_{hg}$. Hence, Δ is fully I-graded. This shows that Δ is a well defined central fully I-graded H-algebra over F. Furthermore, by Lemma 3.3, $\dim_F(\Delta) = |H|$. This completes the proof of the proposition. □

Theorem 3.6. *Let F be a field and G a finite group. Let A and B be equivalent central simple G-algebras over F. Let I be an inertia group of A (and hence of B), and set $H = N_G(I)$. Then, $\Delta(A, I)$ and $\Delta(B, I)$ are isomorphic central fully I-graded H-algebras over F of dimension $|H|$.*

Proof. By Proposition 3.5, $\Delta(A, I)$ and $\Delta(B, I)$ are central fully I-graded H-algebras over F of dimension $|H|$. By Definition 1.5, there exist trivial G-algebras E and E' such that

$$A \otimes E \simeq B \otimes E'$$

as G-algebras. By Proposition 2.5, I is an inertia group of $A \otimes E$ and of $B \otimes E'$. Obviously, $\Delta(A \otimes E, I)$ is isomorphic to $\Delta(B \otimes E', I)$. Hence, it suffices to show that, if A is a central simple G-algebra over F with inertia group I and E is a trivial G-algebra, then $\Delta(A, I)$ is isomorphic to $\Delta(A \otimes E, I)$. By Definition 1.4, we may assume that for some FG-module N, $E = \mathrm{End}_F(N)$. The module structure of N provides a homomorphism

$$\varphi : G \longrightarrow E^{\times}.$$

By the proof of Theorem 2.2, the centers of A and of $A \otimes E$ are canonically G-isomorphic. Hence, we may view the primitive central idempotents of A as also the primitive central idempotents of $A \otimes E$. Furthermore, if $\mathscr{I}$ is as defined in Definition 3.4, then $\mathscr{I}$ also plays the same role for $A \otimes E$. We set $e = \sum_{f \in \mathscr{I}} f$, as in Definition 3.4.

Suppose $h \in I$ and $a \in \Delta(A, I)_h$. Then, $a \in eA$ and, for every $b \in eA$, we have $ab = {}^h b a$. It follows that $a \otimes \rho(h) \in eA \otimes E$, and, for every $b \in eA$, $t \in E$, we have

$$(a \otimes \rho(h))(b \otimes t) = ({}^h b \otimes {}^h t)(a \otimes \rho(h)).$$

As every element of $eA \otimes E$ is a sum of elements of the form $b \otimes t$, it follows that $a \otimes \rho(h) \in \Delta(A \otimes E, I)_h$. We define a map

$$\psi : \Delta(A, I) \longrightarrow \Delta(A \otimes E, I)$$

by setting

$$\psi(a) = a \otimes \rho(h)$$

for $a \in \Delta(A, I)_h$ and $h \in I$, and extending linearly to $\Delta(A, I)$. Then, ψ is a linear map of I-graded vector spaces and it is clearly injective. If $a \in \Delta(A, I)_h$ and $a' \in \Delta(A, I)_{h'}$, h, $h' \in I$, then

$$\psi(aa') = aa' \otimes \rho(hh') = (a \otimes \rho(h))(a' \otimes \rho(h')) = \psi(a)\psi(a'),$$

and, for $g \in H$,

$$^g\psi(a) = {}^g(a \otimes \rho(h)) = {}^ga \otimes \rho({}^gh) = \psi({}^ga).$$

Hence, ψ is an injective morphism of I-graded H-algebras. Since

$$\dim_F(\Delta(A,I)) = |H| = \dim_F(\Delta(A \otimes E, I)),$$

ψ is an isomorphism. This completes the proof of the theorem. $\square$

Finally, we see that the I-centroid behaves well with respect to every extension of the field. If Δ is a central fully I-graded H-algebra over F, then $\Delta \otimes_F K$, where K is some field extension of F, is defined as a graded I-algebra and as an H-module. In fact it is easy to see that $\Delta \otimes_F K$ is a central fully I-graded H-algebra over K.

Proposition 3.7. *Let F be a field, G a finite group, A a central simple G-algebras over F. Let I be an inertia group of A and set $H = N_G(I)$. Let K be an extension field of F. Then, $A \otimes_F K$ is a central simple G-algebra over K which has I as an inertia group, and*

$$\Delta(A \otimes_F K, I) \simeq \Delta(A,I) \otimes_F K.$$

Proof. Set $\bar{A} = A \otimes_F K$. By Proposition 2.7, $\bar{A}$ is a central simple G-algebra over K which has I as an inertia group. Let $\mathscr{I}$ be the set of all primitive central idempotents f of A such that $C_G(fZ(A)) = I$ and $\bar{\mathscr{I}}$ be the set of all primitive central idempotents $\bar{f}$ of $\bar{A}$ such that $C_G(\bar{f}Z(\bar{A})) = I$. By Definition 2.4, $\mathscr{I} \neq \emptyset \neq \bar{\mathscr{I}}$. Set $e = \sum_{f \in \mathscr{I}} f$. Then, e is a central H-invariant idempotent of A. By the proof of Proposition 2.7, $\bar{f} \in \bar{\mathscr{I}}$ if and only if $\bar{f}$ is a primitive central idempotent of $\bar{A}$ and $\bar{f} \subseteq e$. Hence, $\sum_{\bar{f} \in \bar{\mathscr{I}}} \bar{f} = e$. Let $g \in I$. Then, by Definition 3.4,

$$\Delta(A,I)_g = \{a \in Ae\colon \text{for all } b \in Ae,\ ab = {}^gba\},$$

and

$$\Delta(A \otimes_F K, I)_g = \{a \in eA \otimes_F K\colon \text{for all } b \in eA \otimes_F K,\ ab = {}^gba\}.$$

It follows that

$$\Delta(A \otimes_F K, I)_g = \Delta(A,I)_g \otimes_F K.$$

The result follows easily from this. $\square$

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